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Ghosts, knots and Kontsevich integrals

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Abstract

It is shown how ghost propagation in the Hamiltonian formulation of Chern–Simons Field Theory is the physics underlying the Kontsevich integrals: the expectation values of Wilson loops computed to the appropriate order in Perturbation Theory to describe the topology of a knot.

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1. Knot Theory has been always a branch of Low Dimensional Topology strongly influenced by Physics. One of most the recent topics of research into the area is the study of graph cohomology which, inspired by Singularity Theory, provides a framework for understanding the structure of knot invariants through the work of Vassiliev and Kontsevich, see [1] and references quoted therein.

A brief summary of their construction is as follows: extend the space of knots to include immersions of S^1 in an oriented three-manifold M^3 with self-intersections. Given a functional constant on the connected components of the extended knot space, with a grading determined by the number of self-crossings, define the boundary operator acting on the knot invariant V by

$$\nabla V(C_n) = V(C^+_{n-1}) - V(C^-_{n-1}).$$

Here, $V(C_n)$ is the invariant computed for the knot C_n with n self-intersections and the ∇-operation consists in taking the difference of the values of V for the knots $C^{\pm}_{n-1}$, differing from C_n by the fact that the i-sime self-intersection has been solved in a under(over)-crossing. It is obvious how to iterate the process to obtain the ∇^m-operator and because differences are cousins of derivatives it is immediately seen that the Leibnitz rule is satisfied.

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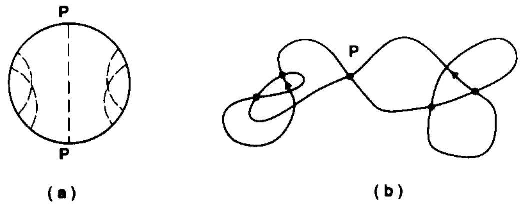

Fig. 1. A cord diagram associated with a knot with self-intersections.

A knot invariant V is called a Vassiliev invariant of type n, an invariant of finite type, if its $(n+1)$th "derivative" vanishes identically: $\nabla^{n+1}V = 0$. The nth coefficients of the Conway, HOMFLY, Jones and Kauffman polynomials are examples, as explained in [1] using the skein relations, of Vassiliev invariants. There is however a general procedure for constructing Vassiliev invariants: to any knot with n self-intersections one can associate a cord diagram of degree n, Fig. 1. A cord diagram is an oriented circle with finitely many cords marked on it modulo orientation-preserving diffeomorphisms of the circle: the degree is the number of cords. A weight system of degree m is a functional on the space of cord diagrams of degree m such that the following properties are satisfied:

(i) If D is a cord diagram with an isolated cord, not intersecting any other cord in D, then $W(C) = 0$.

(ii) If four diagrams A, B, C, D differ as shown in Fig. 2, the dotted areas are the same for all of them, and hence:

$$W(A) - W(C) = W(B) - W(D) \tag{0}$$

Vassiliev invariants are expressed in terms of weight systems. If the functionals on the extended loop and cord diagram spaces take values in $\mathbb{R}$,

(i) Vassiliev invariants of type n are given by the weight system of degree n.

(ii) Vice versa, a weight system of degree n is defined from a Vassiliev invariant of type n.

The strategy is thus to construct a weight system for cord diagrams. There are essentially two aspects:

(a) Algebraic. Lie algebraic invariants are associated with any cord diagram according to the "Feynman Rules" of Fig. 3. The dimension of the representation R running on the external circle, for instance, corresponds to the vacuum diagram. The invariant for the closed diagram with an isolated cord is the quadratic Casimir of the adjoint representation, the representation propagating along the cord and so on. Although it is possible to define trivalent vertices for this representation by means of the Lie bracket, we shall not need to consider this kind of diagrams here.

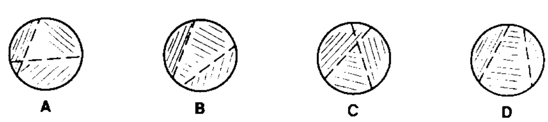

Fig. 2. Cord diagrams to be identified by the four term relation.

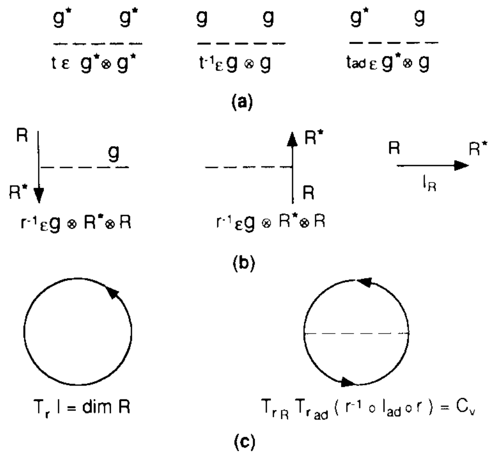

Fig. 3. Feynman rules for a Lie algebra.

(b) Differential topology. The diagrams are also characterized by the locations of the vertices and they will be classified by the topology of the configuration space of n points in $\mathbb{R} \cong \mathbb{C}$. Because a knot is the closure of a braid we shall consider cord diagrams as drawn in Fig. 4a, where the arrows carrying the R representation are directed upwards on a perpendicular axis, labelled by t, to $\mathbb{C}$. We need a complex valued form ω on $\mathbb{C}_1 \times \mathbb{R}_1 \otimes \mathbb{C}_2 \times \mathbb{R}_2$-diagonal to characterize the topology of that space of diagrams such that:

i. it generates a cohomology class: $[\omega] \neq 0$,
ii. there is an involution $\sigma : \mathbb{C}_1 \times \mathbb{R}_2 \leftrightarrow \mathbb{C}_2 \times \mathbb{R}_2$ for which $\omega = -\sigma^* \omega$;
iii. $\omega_{12} \wedge \omega_{23} +$ (cyclic permutations)$= 0$. This is the four term relation (0).

There is a unique solution satisfying those properties, namely,

$$\omega = d \ln |z - \omega| \wedge d\theta(t - s) ,$$

where θ is the Heaviside step function; from this we can construct the formal Knizhnik-Zamolodchikov connection $\Omega = \sum_{i<j} \Omega_{ij} \omega_{ij}$ for the set of diagrams in Fig. 4 contracting with a generating element Ω_{ij} of the algebra defined by the product in Fig. 4b. It is easy to prove that $d\Omega + \Omega \wedge \Omega = 0$, so that Ω is a flat connection with holonomy:

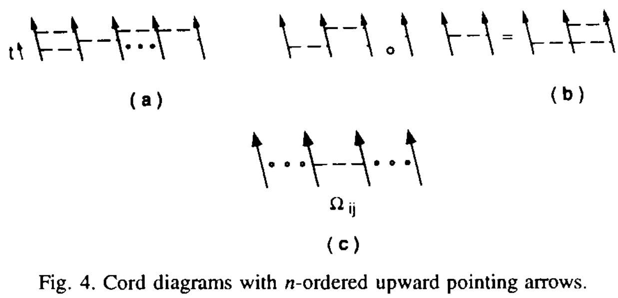

Fig. 4. Cord diagrams with n-ordered upward pointing arrows.

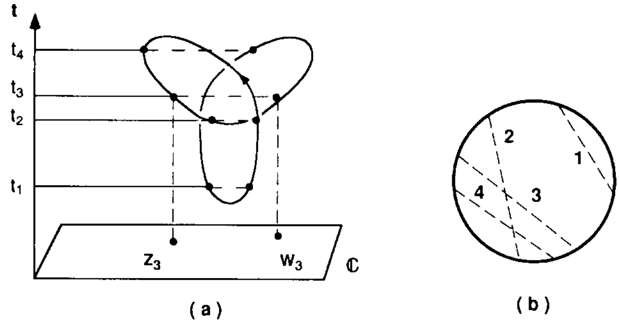

Fig. 5. Morse knot and its cord diagram.

$$\mathrm{Hol}_{\Omega}(B)_a^b = \sum_{m=0} \int_{a \le t_1 \le \cdots \le t_m \le b} B^* \Omega(t_m) \cdots B^* \Omega(t_m).$$

Kontsevich's clever idea was to apply this framework to Morse knots, knots for which t is a Morse function, as in Fig. 5, and define integrals as above, allowing for ascending and descending arrows and including a combinatorial factor D, the Lie algebraic invariant due to the associated cord diagram:

$$K_n(C) = \int_{t_1 \le \cdots \le t_n} \sum_{(z_i, \omega_i)} \left[D\,(-1)^{\lambda_{\uparrow\downarrow}} \bigwedge_{i=1}^{n} \frac{dz_i - d\omega_i}{z_i - \omega_i} \right].$$

The pairings (z_i, ω_i) are taken at values of t which are not critical points of C and $(-1)^{\lambda_{\uparrow\downarrow}}$ accounts for the relative orientation of the corresponding strands. It happens that the Kontsevich integrals provide an explicit expression for a weight system and Vassiliev invariants and they also characterize the different types of Morse knots. Moreover, the combinatorics of the graphs are the same as the combinatorics arising in Perturbative Chern–Simons Field Theory [2]; the challenge is now to understand Vassiliev–Kontsevich invariants in terms of the physics of Chern–Simons systems.

2. To address this issue is the aim of this work. Given the strange partition of $\mathbb{R}^3$ as the cross product of $\mathbb{R}$ by the complex plane $\mathbb{C}$ necessary to define Kontsevich integrals, the natural setting is the Hamiltonian formulation of Chern–Simons Theory. A very practical, and pictorial, tool to handling Perturbation Theory is the use of Feynman Rules and although one of its main advantages is the possibility of covariant formulation, by sticking to the Hamiltonian formalism we are forced to deal with non-covariant Feynman diagrams. In this way the instantaneous propagation of Faddeev–Popov ghosts is the physical representation of the building blocks of the Kontsevich integrals.

We start from the Chern–Simons Lagrangian

$$\mathcal{L}_{\mathrm{CS}} = -\frac{\mu^2}{g_0^2} \varepsilon_{ijk} \operatorname{Tr}[\partial A_j A_k + \tfrac{2}{3} A_i A_j A_k] \tag{1}$$

defined on the space of connections $A_i = A_i^a T^a$ of a SU(N) flat bundle over $\mathbb{R}^3$. Here the T^a are anti-hermitian SU(N) generators with $[T^a, T^b] = f^{abc}T^c$ and $\mathrm{Tr}\, T^aT^b = -\frac{1}{2}\delta^{ab}$ and $\mu_0^2 = \mu^2/g_0^2$, the quotient of the topological mass by the square of the coupling constant, is a dimensionless parameter which for topological reasons will be $k/4\pi$ with k integer. Denoting by Greek indices $\alpha, \beta, \ldots$ spatial components and renaming $a_\alpha = A_\alpha$ for $\alpha = 1, 2$, we write $\mathcal{L}_{\mathrm{CS}}$ in the so-called generalized Hamiltonian form of dynamics,

$$
\begin{aligned}
\mathcal{L}_{\mathrm{CS}} &= -\mu_0^2\, \mathrm{Tr}[\partial_0 a_\alpha \varepsilon_{\alpha\beta} a_\beta - A_0 \varepsilon_{\alpha\beta} F_{\alpha\beta}]\,, \\
F_{\alpha\beta} &= \partial_\alpha a_\beta - \partial_\beta a_\alpha + [a_\alpha, a_\beta]\,.
\end{aligned} \tag{2}
$$

$\Pi_\alpha(\boldsymbol{x}) = \varepsilon_{\alpha\beta} a_\beta(\boldsymbol{x}) \mu_0^2$ and $a_\alpha(\boldsymbol{x})$ are canonically conjugated variables whereas the Lagrange multiplier $A_0(\boldsymbol{x})$ enforces the constraint

$$G(\boldsymbol{x}) = \varepsilon_{\alpha\beta} F_{\alpha\beta}(\boldsymbol{x})\,.$$

By introducing the Poisson brackets

$$\{\Pi_\alpha^a(\boldsymbol{x}), a_\beta^a(\boldsymbol{y})\} = \delta_{\alpha\beta}\delta^{ab}\delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})$$

it is easy to check that the constraints close under the SU(N) algebra

$$\{C^a(\boldsymbol{x}), C^b(\boldsymbol{y})\} = f^{abc} C^c(\boldsymbol{x})\delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})\,.$$

To define the Hamiltonian formalism, even when the Hamiltonian is zero as in this case, we add subsidiary conditions such that $\det \|\{S(\boldsymbol{x}), C(\boldsymbol{y})\}\|$ is different from zero. For the Coulomb gauge

$$S^a(\boldsymbol{x}) = \mu_0^2 \partial_\alpha a_\alpha^a(\boldsymbol{x}) = 0$$

we have:

$$
\begin{aligned}
&\det \|\{S^a(\boldsymbol{x}), C^b(\boldsymbol{y})\}\| \\
&\quad = \det \|[\mu_0^2(-\delta^{ab}\nabla^2 + f^{abc} a_\alpha^c(t, \boldsymbol{x})\partial_\alpha)\delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})]\|\,.
\end{aligned}
$$

3. The Functional Integral defining the quantum dynamics is [3]

$$
\begin{aligned}
\mathbb{Z} = N \int & \mathcal{D}a\mathcal{D}\pi\mathcal{D}A_0 \text{“}\delta(S)\text{”} \det \|\{S, C\}\| \text{“}\delta(\pi - *a)\text{”} \\
&\times \exp\left(-i\mu_0^2 \int d^3x\ \mathrm{Tr}(\partial_0 a_\alpha \pi_\alpha - A_0 C)\right)\,,
\end{aligned}
$$

where we integrate $\mathcal{L}_{\mathrm{CS}}$ in a “normalizing box” of volume L^2T, i.e. we replace $\mathbb{R}^3$ by $T^2 \times S^1$ thus obtaining an infrared cut-off. We shall express in the usual way the determinant of $M^{ab} = \{S^a, C^b\}$ as a Berezin integral on Grassman fields, the Faddeev–Popov ghosts,

$$\det \|M^{ab}(t,z,s,w)\| = \int \mathcal{D}\varphi^a(z)\mathcal{D}\xi^a(z)$$
$$\times \exp\left(i\mu_0^2 \int dt \int ds \int d^2z \int d^2w \, [\varphi^a(z)(M^{ab}(t,s,z,w))\xi^b(z)]\right),$$

where M is

$$M^{ab}(t,s,z,w) = \delta^{(2)}(z-w)(-\delta^{ab}\delta(t-s)\nabla^2 + f^{abc}a^c_\alpha(t,z)\partial^z_\alpha).$$

In Yang–Mills theory one can skip dealing with the troublesome "δ" functionals replacing them by gauge-fixing terms in such a way that in our case, after integrating over A_0, which yields another "$\delta(C)$", $\mathcal{L}_{CS}$ would be replaced by

$$\mathcal{L}_{CS} + \mathcal{L}_{GF} + \mathcal{L}_C = -\mu_0^2 \operatorname{Tr}\left[\varepsilon_{\alpha\beta}a_\alpha\partial_0 a_\beta + \frac{T}{2}\left(\frac{1}{\lambda_1}\partial_\alpha a\partial_\alpha a + \frac{1}{\lambda_2}F_{\alpha\beta}F_{\alpha\beta}\right)\right].$$

There is then one "unphysical" gluon propagating in a non-covariant way, three and four gluon vertices, instantaneous Faddeev–Popov ghosts and a triple gluon–ghost vertex.

In this simpler setting we can solve the equations $C = S = 0$ and take into account the "δ"-functionals by reducing the integration domain to the solution manifold of those equations. In the topological limit when λ_1 and λ_2 tend to zero the generating functional of the Green functions is:

$$\begin{aligned} I[j] = N \int \mathcal{D}q \int \mathcal{D}\varphi\mathcal{D}\xi \, \exp\left\{\tfrac{1}{2}i\mu_0^2 L^2 \int dt \, \varepsilon_{\alpha\beta}q^a_\alpha \dot{q}^a_\beta\right\} \\ \times \exp\left\{i\int dt \int d^2z \, j^a_\alpha(t,z)q^a_\alpha\right\} \\ \times \exp\left\{i\mu_0^2 \int dt \int d^2z \int ds \int d^2w \, \varphi^a(z)M^{ab}[q;t,z,s,w]\xi^b(w)\right\}. \end{aligned} \tag{3}$$

Here the j's are the sources for the q's, which in turn parametrize the space of solutions of $C = S = 0$ in $\mathcal{A}/\mathcal{G}|_{T^2}$, i.e. the moduli space $\mathcal{M}$ of flat connections modulo gauge equivalence; we do not fix, however, the remaining global SU(N) symmetry to keep track of the non-abelianity. The integration domain is therefore the space Maps($S^1, \mathcal{M}$); the integration measure is $\mathcal{D}q = \prod_{a,\alpha,t} dq^a_\alpha(t)$ and this restriction of the integral to the transverse sections to the orbits of the gauge group requires us to include the Jacobian of the map from the Lie algebra of the gauge group to the tangent space to $\mathcal{M}$ in $\mathcal{A}/\mathcal{G}$; this is the job done by the Berezin integration over the ghost fields.

From (3) we read the Feynman rules:

Propagators:

i. Frozen gluon, exclusively time propagation excitation;

$$\Delta^{ab}_{+-}(x_1 - x_2) = \frac{i}{2\mu_0^2 L^2}\delta^{ab}\varepsilon(t_2 - t_1)\delta^{(2)}(z_2 - z_1)\varepsilon_{+-},$$
$$x_i = (t_i, z_i) \in S^1 \times T^2; \quad q^a_\pm(t) = q^a_1(t) \pm i q^a_2(t),$$
$$\varepsilon_{+-} = 1 = -\varepsilon_{-+}, \quad \varepsilon_{++} = \varepsilon_{--} = 0.$$

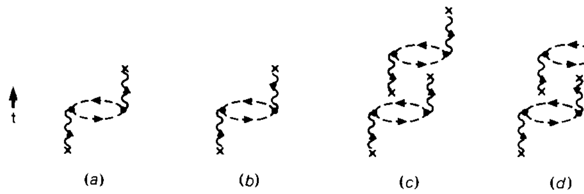

Fig. 6. Graphs for some processes at second and fourth order.

For practical purposes we shall split this into "particle" and "anti-particle" propagation:

$$\Delta^{ab}_{T+-}(x_1 - x_2) = \frac{i}{4\mu_0^2 L^2}\delta^{ab}\theta(t_2 - t_1)\delta^{(2)}(z_2 - z_1)\varepsilon_{+-}\,,$$

$$\Delta^{ab}_{T+-}(x_1 - x_2) = -\frac{i}{4\mu_0^2 L^2}\delta^{ab}\theta(t_1 - t_2)\delta^{(2)}(z_2 - z_1)\varepsilon_{+-}\,.$$

i. Instantaneous ghosts:

$$\Delta^{ab}_{\mathrm{FP}}(x_1 - x_2) = \frac{1}{\mu_0^2 T}\delta(t_1 - t_2)\delta^{ab}\ln|z_1 - z_2|\,.$$

Vertices:

i. Sources: $j^a_\pm(t, z)$;

ii. Gluon–ghost vertex: $\mu_0^2 f^{abc}(q^c_+(t)\partial_+ + q^c_-(t)\partial_-)$.

We shall now compute the contribution to the two-point Green function at second order in Perturbation Theory and to the four-point Green function at fourth order in Perturbation Theory of the graphs of Fig. 6 because of its pertinence to later work. In the first case, Feynman technology gives:

$$\Gamma^{(2)}|^{\uparrow\uparrow}_{2+-} = \frac{c_v\delta^{ab}}{(4\mu_0^2 L^2 T)^2} j^a_+(t_1, z_1)\theta(t - t_1)\varepsilon_{+-}\partial^{z_1}_- \ln|z_1 - z_2|$$
$$\times\varepsilon_{-+}\partial^{z_2}_+ \ln|z_1 - z_2|\theta(t_2 - t) j^b_-(t_2, z_2)\,, \tag{4a}$$

$$\Gamma^{(2)}|^{\uparrow\downarrow}_{2+-} = -\frac{c_v\delta^{ab}}{(4\mu_0^2 L^2 T)^2} j^a_+(t_1, z_1)\theta(t - t_1)\varepsilon_{+-}\partial^{z_1}_- \ln|z_1 - z_2|$$
$$\times\varepsilon_{-+}\partial^{z_2}_+ \ln|z_1 - z_2|\theta(t_2 - t) j^b_-(t_2, z_2)\,, \tag{4b}$$

where c_v is defined by $f^{abc}f^{abd} = c_v\delta^{cd}$. A longer but similar computation for the fourth order graphs yields:

$$\Gamma^{(4)}|^{\uparrow\uparrow\uparrow\uparrow}_{4+-+-} = \Gamma^{(2)}|^{\uparrow\uparrow}_{2+-}\Gamma^{(2)}|^{\uparrow\uparrow}_{2+-}\,, \tag{5a}$$

$$\Gamma^{(4)}|^{\uparrow\uparrow\uparrow\downarrow}_{4+-+-} = \Gamma^{(2)}|^{\uparrow\uparrow}_{2+-}\Gamma^{(2)}|^{\uparrow\downarrow}_{2+-}\,. \tag{5b}$$

4. The central development of this work is the following: Consider an oriented knot C for which the "time" variable t is a Morse function in $\mathbb{R}^3$. This means that C can be parametrized by $z(t)$, adapting the curve to the foliation of $\mathbb{R}^3$ by C_t, in such a

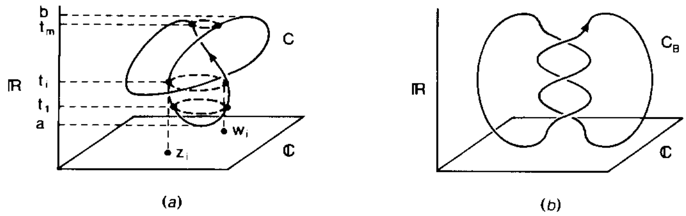

Fig. 7. Morse knot and its corresponding braid.

way that the critical points at values $a < t_1 < t_2 < \cdots < t_m < b$ are isolated and well ordered. The expectation value of the holonomy of a flat connection in $\mathcal{M}$ through C, the Wilson loop, is the key element of the analysis and given by the Functional Integral

$$
\begin{aligned}
\langle W(C)\rangle &= \int \mathcal{D}q \int \mathcal{D}\varphi \mathcal{D}\xi \ \mathrm{Hol}_C(q) \exp\left(i\mu_0^2 L^2 CS[q]\right) \\
&\quad \times \exp(i\mu_0^2 \int d^3x \, \{\varphi^a(z) M^{ab}[q] \xi^b(z)\}) \,, \\
\mathrm{Hol}_C(q) &= \mathrm{Tr}\, P \exp\left(\int_C q\right) \\
&= \mathrm{Tr}\, P \exp\left(\int_C \{dz(t)\, q_+^a(t) + d\bar{z}(t)\, q_-^a(t)\}\right) . \qquad (6)
\end{aligned}
$$

P means path ordering and we compute $\langle W(C)\rangle$ by series expansion:

$$
\begin{aligned}
\langle W(C)\rangle &= 1 + \frac{1}{2!}\left\langle \mathrm{Tr}\, P \int_C q \int_C q \right\rangle \\
&\quad + \frac{1}{4!}\left\langle \mathrm{Tr}\, P \int_C q \int_C q \int_C q \int_C q \right\rangle + \cdots , \qquad (7)
\end{aligned}
$$

the odd terms being zero because the tadpole graph gives no contribution. The goal is to show that $\langle W(C)\rangle$ is a topological invariant independent of continuous deformations of C. Any Morse knot such as C can be deformed by horizontal moves, which keep the critical points fixed, and vertical moves, interchanging or killing critical points, to a braid as shown in Fig. 7. In this form it is easier to compute $\langle W(C)\rangle$ and the strategy will first be to solve the easiest case and then to show independence with respect to those moves. We focus on two strands with one crossing to notice that a non-null contribution to the first term of the expansion in (7) comes from the graphs of Fig. 8 at second order in Perturbation Theory:

$$\frac{1}{2!}\left\langle \operatorname{Tr} P \int_{B_1^P} q \int_{B_1^{P'}} q \right\rangle = -T\frac{(s-t)}{4}\frac{1}{64(\mu_0^2 L^2 T)^2}\cdot\theta(s-t)\cdot\delta^{ab}$$

$$\times \int_{B_1^P}\int_{B_1^{P'}} \left\{ f^{abc}(-1)^{\lambda_{PP'}}\frac{dz(t)-dw(s)}{z(t)-w(s)} \right.$$

$$\left. \wedge\, f^{abc}(-1)^{\lambda_{PP'}}\frac{dz(t)-dw(s)}{z(t)-w(s)} \right\} \tag{8}$$

where $\lambda_{PP'}$ are zero or one depending on the relative orientation of the two strands of B_1: $\lambda_{\uparrow\downarrow} = \lambda_{\downarrow\uparrow} = 1$ and $\lambda_{\uparrow\uparrow} = \lambda_{\downarrow\downarrow} = 0$. Multiplying by the "number of particles" in a box, L^1T/π^3, and taking the limit when s tends to t such that $s - t \cong 4/T$, we obtain

$$\begin{aligned}\langle W(B^1)\rangle_{2PP'} &= \int_{B_1^P}\int_{B_1^{P'}} \left\{ \frac{1}{(2\pi i)}(-1)^{\lambda_{PP'}}\frac{dz-dw}{z-w}f^{abc} \right\} \\ &\quad \wedge \left\{ \frac{1}{2\pi i}(-1)^{\lambda_{PP'}}\frac{dz-dw}{z-w}f^{abc} \right\} \\ &= K^{abc}_{2PP'}(B^1)K^{abc}_{2PP'}(B^1);\end{aligned} \tag{9}$$

the Kontsevich integrals appear as a "square root" of the expectation value of the Wilson loop in second order of Perturbation Theory and the Physics behind it is as follows: By exchanging a ghost pair emitted by q two different points of a braid at the same instant of time are correlated. Integrating along the braid crossings, orientations and combinatoric factors are accounted for and the structure of the braid is described. In fact one needs to consider only half the diagram and this explains the "square root" appearance of (9). In taking the square root, however, we could have chosen a "scalar" decomposition of (9), assigning a weight of $\sqrt{1}$ to each vertex as the result of the combinatorics of the diagram rather than the "tensorial" decomposition where the role of the vertex is more evident. The point is that following Kontsevich we shall measure higher order contributions with respect to $K^{abc}_{2PP'}$, or, equivalently , we shall set $K^{abc}_{2PP'}$ equal to zero, e.g. by contracting with δ^{ab}.

A similar computation for a piece of a braid with two crossings, Fig. 8, yields at fourth order in Perturbation Theory:

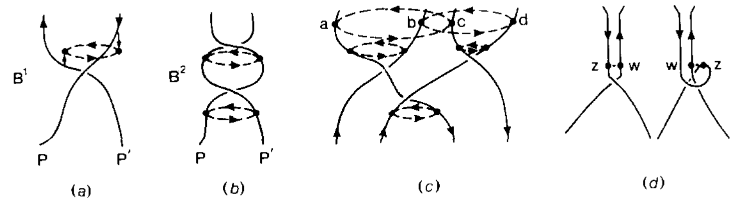

Fig. 8. Diagrams contributing to $\langle W(C_B)\rangle$ at lower orders in Perturbation Theory. Peculiar cases: resolution of a triple crossing, four ways, and close strands.

$$\begin{aligned}\langle W(B^1)\rangle_{\substack{4PP'\\QQ'}} = \int_{B_2^P}\int_{B_2^{P'}}\int_{B_2^Q}\int_{B_2^{Q'}} &\left\{\frac{1}{(2\pi i)^2}(-1)^{\lambda_{PP'}}(-1)^{\lambda_{QQ'}}\frac{dz(t_1)-dw(t_1)}{z(t_1)-w(t_1)}\right.\\ &\left.\wedge\frac{dz(t_2)-dw(t_2)}{z(t_2)-w(t_2)}\cdot c_v\delta^{ab}\delta^{cd}\right\}\\ &\times\left\{\frac{1}{(2\pi i)^2}(-1)^{\lambda_{PP'}}(-1)^{\lambda_{QQ'}}\frac{dz(t_1)-dw(t_1)}{z(t_1)-w(t_1)}\right.\\ &\left.\wedge\frac{dz(t_2)-dw(t_2)}{z(t_2)-w(t_2)}\cdot c_v\delta^{ab}\delta^{cd}\right\}\theta(t_2-t_1)\\ &= K^{abcd}_{\substack{4PQ\\P'Q'}}(B^2)\cdot K^{abcd}_{\substack{4P'Q'\\PQ}}(B^2)\,. \end{aligned}\tag{10}$$

It is interesting to notice here that all non-saturated anti-symmetric indices coming from the f^{abc}'s arising in the vertices and in $\mathrm{Tr}\, T^aT^bT^cT^d$ give no contribution because of the multiplication with the δ^{ab} coming from propagators, and we shall take (10) as the first non-zero Kontsevich integral. At the next, sixth order, in Perturbation Theory, the Feynman Rules speak of three kinds of combinatoric factors: (1) The f^{abc}'s are multiplied by δ^{ab}'s, giving a zero contribution. (2) There is one f^{abc} left, as in $K^{abc}_{2PP'}(B^1)$, which is set to zero for the same reason. (3) There are only δ^{ab} factors giving a non-zero contribution.

We can now give the formula for the general case, a braid B^m with m crossings:

$$\langle W(B^m)\rangle_{2mP_1P'_1\cdots P_mP'_m} = K^{a_1b_1\cdots a_mb_m}_{\substack{2m\,P_1\cdots P^m\\P'_1\cdots P'_m}}(B^m)K^{a_1b_1\cdots a_mb_m}_{\substack{2m\,P_1\cdots P^m\\P'_1\cdots P'_m}}(B^m)\,,\tag{11}$$

$$K^{a_1b_1\cdots a_mb_m}_{\substack{2m\,P_1\cdots P^m\\P'_1\cdots P'_m}}(B^m) = \int_{t_1<t_2<\cdots<t_m}\bigwedge_{i=1}^{m}\frac{c_v}{2\pi i}(-1)^{\lambda_{P_iP'_i}}\delta^{a_ib_i}\cdot\frac{dz(t_i)-dw(t_i)}{z(t_i)-w(t_i)}\,.$$

This extends immediately to a knot C_{B^m} formed by joining the starting and ending points of B^m, Fig. 7b, and provides an invariant for the knot even though the integration is only carried out between the absolute minimum and maximum values of t, modulo $b-a$, a and b at the knot, because the rest of the holonomy is irrelevant.

To prove that $\langle W(C)\rangle_{2m}$ is equal to $\langle W(B^m)\rangle_{2m}$ if C is any Morse knot that can be obtained from C_{B^m} by continuous deformations, we refer to Ref. [1]. We briefly comment on a physical interpretation of the arguments in [1]. If A_n is the subspace of grade n in the space of extended knots the splitting given by the boundary operator

$$\nabla W(C_n) = W(C^+_{n-1}) - W(C^-_{n-1}),$$

where $W(C_n)$ is the value of some functional on A_n, the element C_n is associated with the ghost propagation and due to the fact that an m step descent is required to get A_0 from A_m, we find the natural invariant in the $2m$th order of Perturbation Theory. New contributions arise at any $2mn$th order and we obtain a series expansion of the Witten–Jones invariant answering in the affirmative a question posed in [1] about the possibility of approximating this invariant in terms of Vassiliev invariants:

$$WJ_m(C_{B^m}) = \langle W(C_{B^m}) \rangle_{2m\, {}^{p_1\cdots p_m}_{p'_1\cdots p'_m}} , \tag{12}$$

$$WJ(C_{B^m}) = \sum_{n=0}^{\infty} \frac{1}{n!} [WJ_m(C_{B^m})]^n = \exp\{iWJ_m(C_{B^m})\}.$$

To deform C_{B^m} to any C also in A_m and still a Morse knot two kinds of moves are allowed:

(a) Horizontal moves where the critical points occur for the same values of t. Mathematically, it is shown in [1] that $K_{2m}(C_{B^m}) = K_{2m}(C)$ using the fact that:

$$\Omega_{m,m} = \sum_{1\le i\le j\le 2m} s_i s_j \Omega^G_{ij} \omega_{ij} , \tag{13}$$

$$\Omega^G_{ij} = \tfrac{1}{2}[q_{+i}, q_{+j}] , \quad \omega_{ij} = d \log|z_i - z_j| , \quad s_i s_j = (-1)^{\lambda_{p_i p_j}} , \tag{14}$$

is a flat connection. Physically Ω^G_{ij} comes from the frozen gluons emitting the ghost pair at t_i and t_j, ω_{ij} is due to the infinite speed propagation of the ghosts. No wonder the invariance under horizontal deformations, they do not notice the distance!.

(b) Vertical moves. There are two possibilities for obtaining apparently distinct Morse knots. In the first case some of the minima and maxima are exchanged. The Kontsevich integrals are the same. Check: 1. Deform the initial knot to another where the critical points to be exchanged are very sharp by horizontal moves. 2. Compute K_{2m} to see that the contribution of the needles is negligible. 3. Reject the needles. This process does not change the integrals. In the second case consecutive maxima and minima are killed by stretching the piece of the curve containing them. Because the number of critical points cannot be changed a correction must be made: let C_r be a Morse knot with r-maxima in A_m and C_0 the Morse knot in A_0 with two maxima. The quantity

$$\tilde{K}_{2m}(C_r) = K_{2m}(C_r)/K_{2m}(C_0)$$

is also invariant under these moves.

It would be interesting to prove that $\nabla^{m+1}K_{2m}(C) = 0$ for $C \in A_m$. Consider

$$\nabla^m K_{2m}(C) = K_{2m}(C^+) - K_{2m}(C^-),$$

where $C^\pm \in A_{m-1}$ and are identical to C except that one self-intersection has been solved in an over (under)-crossing. Horizontal deformation freedom allow us to choose $C^\pm$ as in Fig. 8d. One thus has that:

$$K_{2m\, {}^{p_1\cdots p_m}_{p'_1\cdots p'_m}}(C^+) - K_{2m\, {}^{p_1\cdots p_m}_{p'_1\cdots p'_m}}(C^-) = \int_{|z-w|<\varepsilon} \frac{dz - dw}{z - w} K ,$$

which is $2\pi i K$ because of the residue theorem. K is a constant, the contribution of the rest of the diagram, while z and w are very close and exchanged, and the proposition follows.

The case when $t_i = t_{i+1}$ is more involved. It corresponds to a triple self-intersection, which can be solved as drawn in the diagrams of Fig. 8c in four different ways:

[1,2,3,4], [2,1,3,4] and [1,2,4,3] for $[a,b,c,d]$ plus the simplest diagram not labelled in the figure. Therefore the four term relation (0) of Vassiliev Theory is no more than the identical contribution of the diagrams corresponding to ghost exchange between two pairs of strands, in which a triple crossing is solved, either intertwined or not in this Field Theoretical setting.

5. We have not touched upon the problem of divergences: when z gets too close to w infinities arises. Fortunately these are harmless for Knot Theory because we have set the dangerous graphs down to zero. In a broader context we can address the issue as a Renormalization matter and for the sake of completeness we sketch a proof of the renormalizability of Chern–Simons theory in this Hamiltonian form. We focus on the effective potential, the functional generatrix of the one-particle irreducible graphs. For a constant external field of the form $\bar{q}^a_\alpha(t) = \varepsilon_{\alpha\alpha_1}\bar{q}^a_{\alpha_1}$; this is up to one loop order in the $1/\mu_0^2$-expansion:

$$V_{\text{eff}}[\sigma^2] = 2\mu_0^2\sigma^2 + \int \frac{d^2p}{(2\pi^2)} \cdot \sum_{n=1}^{\infty} \frac{1}{2n}\left(c_v \frac{\bar{q}^a_\alpha \bar{q}_\alpha}{|p|^2}\right),$$

where σ^2 is $\sigma^2 = \bar{q}^a_\alpha \bar{q}^a_\alpha$, The first term is due to the Chern–Simons action of the external field acted upon the Green function of the operator $\sqrt{-d^2/dt^2}$,

$$\bar{q}^a_\alpha(t) = \varepsilon_{\alpha\alpha_1}\bar{q}^a_{\alpha_1} \cdot |t-T|^{1/2}.$$

The second term is the sum of the diagrams with n vertex insertions of $\varepsilon_{\alpha\alpha_1}\bar{q}_{\alpha_1}$ in momentum space. Subtracting the contribution of $\sigma^2 = 0$, which is infrared divergent, we have:

$$V^R_{\text{eff}}[\sigma^2] = 2\mu_0^2\sigma^2 + \int \frac{d^2p}{(2\pi)^2} \ln\left(1 + c_v \frac{\sigma^2}{|p|^2}\right).$$

However, one still has the need of an ultraviolet cut-off to find the regularized effective potential

$$V^R_{\text{eff}}[\sigma^2] = 2\mu_0^2\sigma^2 + c_v \frac{\sigma^2}{4\pi}\left(\ln \frac{c_v\sigma^2}{\Lambda^2} - 1\right).$$

The coupling constant is defined at an arbitrary renormalization point

$$\left.\frac{d^2V^R}{d\sigma^2}\right|_{M^2} = 4\mu_R^2 = 4\mu_0^2 + \frac{c_v}{2\pi} \ln \frac{c_v M^2}{\Lambda^2} + \frac{c_v}{\pi},$$

and, choosing the bare coupling constant to be

$$\mu_0^2 = \frac{k}{\pi} + \frac{c_v}{2\pi} \ln \frac{c_v M^2}{\Lambda^2}, \quad k \in \mathbb{Z},$$

everything is finite at this order in the loop expansion:

$$V^R_{\text{eff}}[\sigma^2] = k\frac{\sigma^2}{\pi} + c_v \frac{\sigma^2}{4\pi}\left(\ln \frac{c_v\sigma^2}{M^2} - 3\right).$$

The famous shift of Chern–Simons theory is disguised in the finite renormalization:

$$\mu_R^2 = \frac{k + c_v}{4\pi} = \left.\frac{d^2 V_{\text{eff}}^R}{d\sigma^2}\right|_{M^2}.$$

From the very beginning it was obvious that our model is a theory of fermions interacting with an external field. No wonder that we met the effective potential of the Gross–Neveu model. Also, according to the results of Ref. [4], higher orders of loop expansion do not alter our conclusion about the finiteness of the model, nor are other corrections to K needed.

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